Kumaraswamy Alpha Power Weibull Distribution: Properties and Applications

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Abstract

We introduce the Kumaraswamy alpha power-G (KuAP-G) family which extends the alpha power transform class (Mahdavi and Kundu, 2015) and some other families. We consider the Weibull as baseline for the KuAP family and generate Kumaraswamy alpha power Weibull distribution which has symmetrical, right-skewed, left-skewed and reversed-J shaped densities, and decreasing, increasing, bathtub, upside-down bathtub, J shaped and reversed-J shaped hazard rates. The importance of the new distribution comes from its ability to model monotone and non-monotone failure rate functions, which are quite common in reliability studies. We derive some basic properties of the new. The maximum likelihood estimation method is used to evaluate the parameters and the observed information matrix is determined. We illustrate the performance of the proposed family of distributions by means of two real data sets.

Keywords: Alpha Power family, Kumaraswamy family; Maximum likelihood estimation; Weibull distribution.

Introduction

Recently, Mahdavi and Kundu (2015) proposed a new class of distributions called the alpha power transformation (APT) family. For any baseline cumulative distribution function (CDF) $G(x)$, Mahdavi and Kundu (2015) defined the CDF of the APT family (for $x \in \mathcal{R}$) by

$$H_{APT}(x)=\begin{cases}\dfrac{\alpha^{G(x)}-1}{\alpha-1} & if\ \alpha>0,\alpha\neq 1\\ G(x) & if\ \alpha=1.\end{cases} \tag{1}$$

and the corresponding probability density function (PDF) as

$$h_{APT}(x)=\begin{cases}\dfrac{\ln\alpha}{\alpha-1}g(x)\alpha^{G(x)} & if\ \alpha>0,\alpha\neq 1\\ g(x) & if\ \alpha=1.\end{cases} \quad (2)$$

In this paper, we define and study a new family of distributions with three extra shape parameters to provide more flexibility to the generated class. In fact, based on the Kumaraswamy-G (K-G) family proposed by Cordeiro and de Castro (2011), we construct a new class called the Kumaraswamy alpha power-G (KuAP-G) family and provide a comprehensive description of some of its mathematical properties. We hope that the new model will attract wider applications in reliability, engineering and other areas of research.

Consider the CDF and PDF of a given random variable namely $G(x)$ and $g(x)$. Then, the CDFand PDF of the K-G family are, respectively, given by

$$F(x)=1-[1-G(x)^{a}]^{b},\ a,b>0 \quad (3)$$

and

$$f(x)=abg(x)G(x)^{a-1}[1-G(x)^{a}]^{b-1},\ a,b>0. \quad (4)$$

Afify et al. (2017) considered the complementary Weibull geometric distribution as a baseline distribution in (3) and introduced the Kumaraswamy complementary Weibull geometric distribution. Afify et al. (2016a) introduced a new method for generating distributions based on the K-G family called the Kumaraswamy transmuted-G family.

We define a new KuAP-G family by taking the APT CDF (1) as the baseline CDF in Equation (3). For a given baseline distribution G, the KuAP-G distribution can be used effectively for real data analysis. We discuss some general mathematical properties of the new family. We consider the Weibull as baseline for the KAP-G family and generate a four-parameter KuAP-Weibull (KuAP-W) distribution, which has several desirable properties.

Two real data sets have been analyzed for illustrative purposes. Other motivations for the KuAP-W distribution are: (i) it contains some lifetime sub-models such as the Weibull, Kumaraswamy Weibull by Cordeiro et al. (2010) and exponentiated Weibull Mudholkar et al (1995), among others; (ii) it is capable of modeling monotonically decreasing, increasing, bathtub, upside down bathtub and reversed-J hazard rates; (iii) it can be viewed as a suitable model for fitting skewed data which may not be properly fitted by other common distributions and can also be used in a variety of problems in different areas such as financial, industrial reliability and survival analysis; and (iv) Two applications to real data prove empirically that it compares well with eight other competing lifetime distributions.

The rest of this paper is organized as follows: in Section 2, we define the KuAP-G distribution and some special cases are presented. In Section 3, we study the new KuAP-W distribution. Some of its structural properties including quantile function, moments, moment generating function, residual and reversed residual lifes and order statistics are derived in Section 4. The maximum likelihood estimates (MLEs) of the model parameter are obtained in Section 5. In Section 6, the analysis of two real data sets have been presented to illustrate the potentiality of the new model. Finally, we provide some conclusions in Section 7.

2. The KuAP-G family

The CDF of the KuAP-G family is obtained by replacing $G(x)$ in Equation (3) by $H_{APT}(x)$ of the APT class given by (1). We have

$$F(x)=\begin{cases}1-\left\{1-\left[\dfrac{\alpha^{G(x)}-1}{\alpha-1}\right]^{a}\right\}^{b} & \text{if } \alpha,a,b>0,\alpha\neq 1\\ 1-\{1-G(x)^{a}\}^{b} & \text{if } \alpha=1.\end{cases} \tag{5}$$

The KAP-G PDF can be expressed as

$$f(x)=\begin{cases}\dfrac{ab\ln(\alpha)}{\alpha-1}g(x)\alpha^{G(x)}\left[\dfrac{\alpha^{G(x)}-1}{\alpha-1}\right]^{a-1}\left\{1-\left[\dfrac{\alpha^{G(x)}-1}{\alpha-1}\right]^{a}\right\}^{b-1} & \text{if } \alpha,a,b>0,\alpha\neq 1\\ ab\,g(x)G(x)^{a-1}[1-G(x)^{a}]^{b-1} & \text{if } \alpha=1.\end{cases} \tag{6}$$

The quantile function (QF) of X, $Q(p)=F^{-1}(p)$, can be obtained by inverting (5) numerically and it takes the form

$$Q_{KAP}(p)=G^{-1}\left(\frac{\log\left\{1+(\alpha-1)[1-(1-p)^{1/b}]^{1/a}\right\}}{\log(\alpha)}\right),\alpha\neq 1.$$

A random sample of size n from (5) can be obtained (for $\alpha\neq 1$), based on the above equation, as $X_i=Q_{KAP}(U_i)$, where $U_i\sim$Uniform$(0,1)$, $i=1,\ldots,n$.

The new KAP-G class contains some special cases which are listed in Table 1.

Table 1: Sub-families of the KAP-G family

α	a	b	Reduced family	Authors
α	a	1	Exponentiated alpha power-G (EAP-G)	New
1	a	b	Kumaraswamy-G (K-G)	Cordeiro and de Castro (2011)
α	1	1	Alpha power-G (AP-G)	Mahdavi and Kundu (2015)
1	a	1	Exponentiated-G (E-G)	Gupta et al. (1998)

By using the generalized binomial expansion and the power series,

$$\alpha^z = \sum_{k=0}^{\infty} \frac{(\ln\alpha)^k z^k}{k!},$$

$$(1-z)^q = \sum_{k=0}^{\infty} \binom{q}{k} (-1)^k z^k, q > 0.$$

we obtain a useful linear representation for the PDF (6) (for $\alpha > 0, \alpha \neq 1$) as

$$f(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(\alpha-1)^{a(i+1)}} \binom{b-1}{i} \left[\alpha^{G(x)} - 1\right]^{a(i+1)-1}.$$

Hence

$$\left[\alpha^{G(x)} - 1\right]^{a(i+1)-1} = \alpha^{[a(i+1)-1]G(x)} \sum_{j=0}^{\infty} (-1)^j \binom{a(i+1)-1}{j} \alpha^{-jG(x)}$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{a(i+1)-1}{j} \alpha^{[a(i+1)-j-1]G(x)}.$$

Then, we can write

$$f(x) = ab\ln(\alpha)\, g(x) \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{(\alpha-1)^{a(i+1)+1}} \binom{b-1}{i} \binom{a(i+1)-1}{j} \alpha^{[a(i+1)-j]G(x)}.$$

Using the power series

$$\alpha^z = \sum_{k=0}^{\infty} \frac{(\ln\alpha)^k z^k}{k!},$$

the PDF of the KAP-G class reduces to

$$f(x) = ab \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j}[a(i+1)-j]^k}{k!\,(\alpha-1)^{a(i+1)+1}[\ln(\alpha)]^{-k-1}} g(x) G(x)^k \binom{b-1}{i} \binom{a(i+1)-1}{j}.$$

Then, we have

$$f(x) = \sum_{k=0}^{\infty} \delta_k \, h_{k+1}(x),$$

where $h_{k+1}(x) = (k+1)g(x)G(x)^k$is the exponentiated-G density with power parameter and$(k+1) > 0$ and

$$\delta_k = ab \sum_{i,j=0}^{\infty} \delta_k \frac{(-1)^{i+j}[\ln(\alpha)]^{k+1}[a(i+1)-j]^k}{(k+1)!\,(\alpha-1)^{a(i+1)+1}} \binom{b-1}{i} \binom{a(i+1)-1}{j}.$$

3. The KuAP-W distribution

The Weibull distribution is a popular life time distribution in reliability theory. Numerous articles have been written demonstrating applications of the Weibull distribution in biological, medical, engineering, meteorology etc. In the last few years, several researchers have developed various extensions and generalized forms of the Weibull distribution to model various types of data. Among these, Mudholkar et al. (1995) and Mudholkar et al. (1996) introduced and studied the exponentiated Weibull distribution to analyze bathtub failure data by adding an extra shape parameter to the Weibull distribution.

Further, Xie and Lai (1995) proposed the additive Weibull distribution, the Weibull extension distribution proposed by Xie et al. (2002), generalized modified Weibull distribution introduced by Jalmar et al. (2008), Kumaraswamy Weibull distribution proposed by Cordeiro et al. (2010), Kumaraswamy generalized gamma distribution introduced by de Castro et al. (2011), Kumaraswamy generalized half-normal distribution proposed by Cordeiro et al. (2012), Kumaraswamy Pareto distribution introduced by Bourguignon et al. (2013), the exponential-Weibull distribution proposed by Cordeiro et al. (2014), Nassar et al. (2017) introduced the alpha logarithmic transformed Weibull distribution and Cordeiro et al. (2017) introduced a new three-parameter lifetime model called the Lindley Weibull distribution.

The random variable T is said to have a two-parameter Weibull (W) distribution with the scale parameter $\lambda > 0$ and shape parameter $\beta > 0$, if the CDF of $t > 0$ is

$$F_W(t;\lambda,\beta) = 1 - e^{-\lambda t^{\beta}}, \lambda,\beta > 0 \qquad (7)$$

and the corresponding PDF is

$$f_W(t;\lambda,\beta)=\lambda\beta t^{\beta-1}e^{-\lambda t^{\beta}},\ \lambda,\beta>0. \tag{8}$$

Inserting (7) in Equation (5), the CDF of the KAPW distribution is

$$F(x,\varphi)=\begin{cases}1-\left[1-\left(\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{b} & if\, \alpha\neq 1\\ 1-\left[1-\left(1-e^{-\lambda x^{\beta}}\right)^{a}\right]^{b} & if\, \alpha=1.\end{cases} \tag{9}$$

The PDF corresponding to (9) is

$$f(x,\varphi)=\begin{cases}\dfrac{a\,b\,\lambda\,\beta(\ln\alpha)}{\alpha-1}x^{\beta-1}e^{-\lambda x^{\beta}}\alpha^{1-e^{-\lambda x^{\beta}}}\left(\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a-1}\left[1-\left(\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{b-1} & if\ \alpha\neq 1\\ ab\lambda\beta e^{-\lambda x^{\beta}}(1-e^{-\lambda x^{\beta}})^{a-1}[1-(1-e^{-\lambda x^{\beta}})^{a}]^{b-1} & if\ \alpha=1\end{cases} \tag{10}$$

The survival function and the hazard rate function (HRF) of X are, respectively, given by

$$S(x,\varphi)=\begin{cases}\left[1-\left(\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{b} & if\ \alpha\neq 1\\ \left[1-\left(1-e^{-\lambda x^{\beta}}\right)^{a}\right]^{b} & if\ \alpha=1\end{cases}$$

and

$$h(x,\varphi)=\begin{cases}\dfrac{a\,b\lambda\beta\log\alpha}{\alpha-1}x^{\beta-1}e^{-\lambda x^{\beta}}\alpha^{1-e^{-\lambda x^{\beta}}}\left(\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a-1}\left[1-\left(\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{-1} & if\,\alpha\neq 1\\ a\,b\lambda e^{-\lambda x^{\beta}}(1-e^{-\lambda x^{\beta}})^{a-1}[1-(1-e^{-\lambda x^{\beta}})^{a}]^{-1} & if\,\alpha=1.\end{cases}$$

Table 2 lists seventeen important special models of the new distribution.

Figures 1 and 2 display some plots of the KuAP-W density function for some selected parameter values. Plots of the HRF of the KuAP-W distribution for some selected parameter values are given in Figure 3.

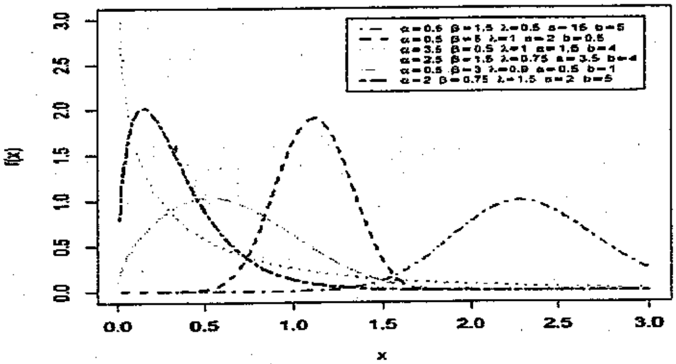

Figure 1: Plots of the PDF of the KAPW distribution for various values of parameters

Table 2: Sub-models of the KAPW distribution

α	a	b	λ	β	Reduced model	Authors
α	a	b	λ	2	KAP-Rayleigh	New
α	a	b	λ	1	KAP-exponential	New
1	a	b	λ	β	K-Weibull	Cordeiro et al. (2010)
1	a	b	λ	2	K-Rayleigh	New
1	a	b	λ	1	K-exponential	New
1	a	1	λ	β	E-Weibull	Mudholkar et al (1995).
1	a	1	λ	2	E-Rayleigh	New
1	a	1	λ	1	E-exponential	Gupta and Kundu (2001).
α	a	1	λ	β	EAP-Weibull	New
α	a	1	λ	2	EAP-Rayleigh	New
α	a	1	λ	1	EAP-exponential	New
α	1	1	λ	β	AP-Weibull	Nassar and Mead (2017).
α	1	1	λ	2	AP-Rayleigh	Malik and Ahmad (2017)
α	1	1	λ	1	AP-exponential	Mahdavi and Kundu (2015)
1	1	1	λ	β	Weibull	Known
1	1	1	λ	2	Rayleigh	Known
1	1	1	λ	1	Exponential	Known

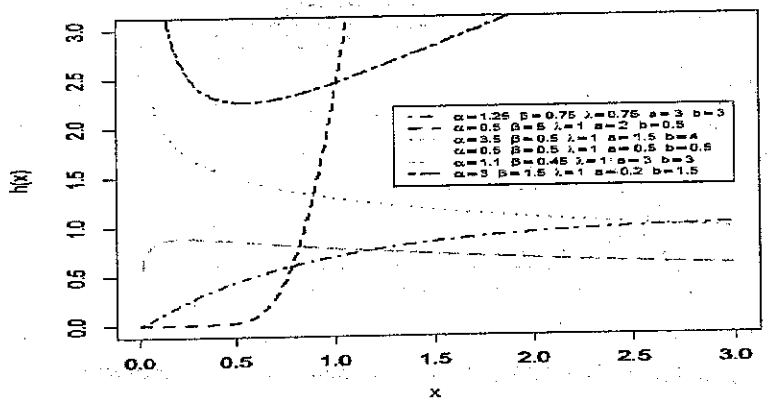

Figure 2: Plots of the HRF of the KAPW distribution for various values of parameters

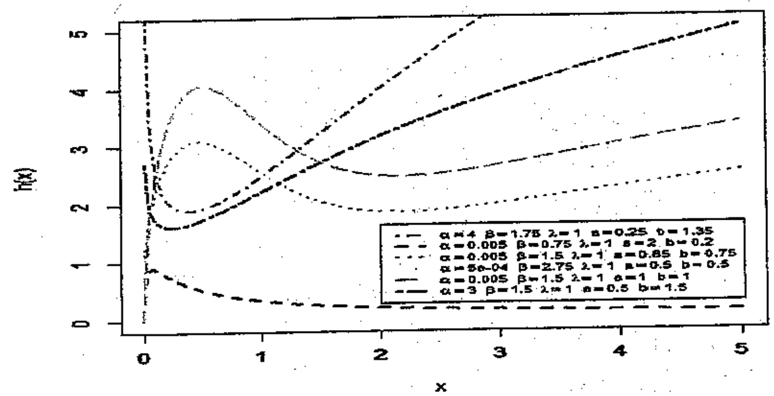

Figure 3: Plots of the HRF of the KAPW distribution for various values of parameters

4. Properties of KAPW distribution

4.1 Linear representation

The KuAP-W density function can be expressed as a linear mixture of W densities

$$f(x) = ab \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j}[a(i+1)-j]^k}{k!\,(\alpha-1)^{a(i+1)+1}[\ln(\alpha)]^{-k-1}} \left(\lambda\beta x^{\beta-1} e^{-\lambda x^{\beta}}\right)$$

$$\times \left(1-e^{-\lambda x^{\beta}}\right)^{k} \binom{b-1}{i} \binom{a(i+1)-1}{j}.$$

Applying the binomial expansion to $\left(1-e^{-\lambda x^{\beta}}\right)^{k}$, we can write

$$f(x) = ab \sum_{i,j,k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^{i+j+m}[a(i+1)-j]^{k}}{k!\,(\alpha-1)^{a(i+1)+1}[\ln(\alpha)]^{-k-1}} \lambda\beta x^{\beta-1} e^{-(m+1)\lambda x^{\beta}} \binom{b-1}{i} \binom{a(i+1)-1}{j} \binom{k}{m}.$$

Then, we have

$$f(x) = \sum_{m=0}^{\infty} d_m\, g_{m+1}(x;\beta,(m+1)\lambda), \tag{11}$$

where $g_{m+1}(x;\beta,(m+1)\lambda)$ is the W PDF with shape parameter β and scale parameter $(m+1)\lambda$, and d_m is the constant term

$$d_m = \sum_{i,j,=0}^{\infty} \sum_{k=m}^{\infty} \frac{(-1)^{i+j+m} ab[a(i+1)-j]^{k}}{k!\,(m+1)(\alpha-1)^{a(i+1)+1}[\ln(\alpha)]^{-k-1}} \binom{b-1}{i} \binom{a(i+1)-1}{j} \binom{k}{m}.$$

Equation (11) shows that the KAPW pdf can be written as a linear mixture of W densities. Then, several of its properties can be obtained from those of the W distribution.

Let Y be a random variable having the W distribution in (7). Hence, the rth ordinary and incomplete moments of Y are, respectively, expressed as

$$\mu'_{r,Y} = \lambda^{-\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta}+1\right), \quad \varphi_{r,Y}(t) = \lambda^{-\frac{r}{\beta}} \gamma\left(\frac{r}{\beta}+1, \lambda t^{\beta}\right),$$

where the lower incomplete gamma function is defined by $\gamma(a,b) = \int_0^b x^{a-1} e^{-x} dx$.

4.2 Quantile Function

Using Equation (9), the KuAP-W distribution can be easily simulated by

$$Y = -\frac{1}{\lambda} \ln \left\{ \frac{\ln\left[\alpha/(\alpha-1)[1-(1-U)^{1/b}]^{1/a}+1\right]}{\ln\alpha} \right\}^{\frac{1}{\beta}},$$

where U follows uniform $(0,1)$ distribution. The p-th quantile function of KAPW distribution is given by

$$Y_p = -\frac{1}{\lambda}\ln\left\{\frac{\ln\left[\alpha/(\alpha-1)[1-(1-p)^{1/b}]^{1/a}+1\right]}{\ln\alpha}\right\}^{\frac{1}{\beta}}.$$

4.3 Moments

The rth moment of X can be obtained from Equation (11) as

$$E(X^r) = \sum_{m=0}^{\infty} d_m\,[(m+1)\lambda]^{-\frac{r}{\beta}}\Gamma\left(\frac{r}{\beta}+1\right).$$

The rth incomplete moment of X is given by $\varphi_r(t) = \int_0^t x^r f(x)dx$. It follows from Equation (11)

$$\varphi_r(t) = \sum_{m=0}^{\infty} d_m \int_0^t x^r g_{m+1}(x)dx,$$

and then, we obtain

$$\varphi_r(t) = \sum_{m=0}^{\infty} d_m[(m+1)\lambda]^{-\frac{r}{\beta}}\gamma\left(\frac{r}{\beta}+1,[(m+1)\lambda]t^{\beta}\right).$$

The first incomplete moment of X follows from the last equation by setting $r = 1$. It can be applied to obtain mean deviations, Bonferroni and Lorenz curves, mean residual and waiting times and totality of deviations from the mean and median.

4.4 Moment generating function

We now provide the moment generating function (MGF) of the W model as derived by Nadarajah et al. (2013). We can write the MGF of Y as

$$M_Y(t;\beta,\lambda) = \beta\lambda\int_0^{\infty} e^{tx}x^{\beta-1}e^{-\lambda x^{\beta}}dx.$$

By expanding e^{tx}andcalculating the integral, we have

$$M_Y(t;\beta,\lambda) = \sum_{k=0}^{\infty}\frac{\left(t/\lambda^{1/\beta}\right)^k}{k!}\Gamma\left(\frac{k}{\beta}+1\right).$$

Using the Wright generalized hypergeometric function defined by

$$ {}_p\Psi_q\left[\begin{matrix}(\lambda_1,A_1),\dots,(\lambda_p,A_p)\\(\beta_1,B_1),\dots,(\beta_p,B_p)\end{matrix};x\right]=\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\lambda_j+A_jm)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jm)}\frac{x^m}{m!}. $$

Hence, we can write the MGF of Y as

$$ M_Y(t;\beta,\lambda)=\ {}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};t/\lambda^{1/\beta}\right]. $$

Combining the above expression and Equation (11), the MGF of X can be expressed as

$$ M_X(t;\beta,\lambda)=\sum_{m=0}^{\infty}d_m\ {}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};t/[(m+1)\lambda]^{1/\beta}\right]. $$

4.5 Residual and reversed residual Lifes

For $n=1,2,\dots$ and $t>0$, the nth moment of the residual life of X is given by

$$ m_n(t)=\frac{1}{1-F(t)}\int_t^{\infty}(x-t)^n dF(x). $$

Using Equation (11), we can write

$$ m_n(t)=\frac{1}{\bar{F}(t)}\sum_{m=0}^{\infty}\sum_{i=0}^{n}\frac{(n+1)_i t^{n-i}}{(-1)^{n-i}i!}d_m[(m+1)\lambda]^{-\frac{r}{\beta}}\gamma\left(\frac{r}{\beta}+1,[(m+1)\lambda]z^{\beta}\right), $$

where $\rho_i=\Gamma(\rho+1)/\Gamma(\rho-i+1)$ is the falling factorial.

The mean residual life (MRL) function of X follows by setting $n=1$ in the last equation. It represents the expected additional life length for a unit which is alive at age x. The MRL is also known as the life expectancy at age x.

For $n=1,2,\dots$ and $t>0$, the nth moment of the reversed residual life of X is given by

$$ M_n(t)=\frac{1}{F(t)}\int_0^{t}(t-x)^n dF(x). $$

Then, we can write

$$ M_n(t)=\frac{1}{F(t)}\sum_{i=0}^{n}\sum_{m=0}^{\infty}\frac{(n+1)_i t^{n-i}}{(-1)^{i}i!}d_m[(m+1)\lambda]^{-\frac{r}{\beta}}\gamma\left(\frac{r}{\beta}+1,[(m+1)\lambda]z^{\beta}\right). $$

The mean inactivity time (MIT) of X follows by setting $n=1$ in the above equation. It represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,x)$. The MIT is also called the mean reversed residual life function.

4.6 Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n, and let $X_{i:n}$ denote the ith order statistic, then, the PDF of $X_{i:n}$, say $f_{i:n}(x)$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \tag{12}$$

Substituting (9) and (10) in (12) we can write $f_{i:n}(x)$ as

$$f_{i:n}(x) = \frac{a\,b\,\lambda\beta\ln\alpha}{\beta(i,n-i+1)} x^{\beta-1} e^{-\lambda x^{\beta}} \alpha^{1-e^{-\lambda x^{\beta}}} \left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a-1}$$

$$\times \left[1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{b(n-i+1)-1} \left[1-\left(1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right)^{b}\right]^{i-1} \tag{13}$$

Using the binomial expansion, then $f_{i:n}(x)$ is given by

$$f_{i:n}(x) = \sum_{k=0}^{i-1} \sum_{j,m,s=0}^{\infty} \sum_{l=0}^{s} \frac{(-1)^{k+j+m} a b (\ln\alpha)^{s+1}\left[a(j+1)-m\right]^{s}}{\beta(i,n-i+1)s!(\alpha-1)^{a(j+1)}} \binom{i-1}{k} \binom{b(n+k-i+1)-1}{j}$$
$$\times \binom{a(j+1)-1}{m} \binom{s}{l} \lambda\beta x^{\beta-1} e^{-(l+1)\lambda x^{\beta}}.$$

The above equation can be rewritten as

$$f_{i:n}(x) = \sum_{l=0}^{\infty} d_l\, g_{l+1}(x;\beta,(l+1)\lambda),$$

where $g_{l+1}(x;\beta,(l+1)\lambda)$ as before, is W density with parameters β and $(l+1)\lambda$, and

$$d_l = \sum_{k=0}^{i-1} \sum_{j,m=0}^{\infty} \sum_{s=l}^{\infty} \frac{a b (\ln\alpha)^{s+1}\left[a(j+1)-m\right]^{s}}{\beta(i,n-i+1)s!(\alpha-1)^{a(j+1)}} \binom{i-1}{k} \binom{b(n+k-i+1)-1}{j} \binom{a(j+1)-1}{m} \binom{s}{l} \frac{(-1)^{k+j+m}}{(l+1)}.$$

The qth moments of $X_{i:n}$ can be expressed as

$$E(X_{i:n}^{q}) = \sum_{l=0}^{\infty} d_l\, [(l+1)\lambda]^{-\frac{r}{\beta}} \Gamma\left(\frac{r}{\beta}+1\right).$$

5. Estimation

Let $x_1, x_2, ..., x_n$ be a random sample from KAPW distribution then the logarithm of the likelihood function (ℓ), becomes

$$\ell = n[\ln a + \ln b + \ln\lambda + \ln\beta + \ln\alpha] + n\ln\left(\frac{\ln\alpha}{\alpha-1}\right) + (\beta-1)\sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}x_i^{\beta} - \ln\alpha\sum_{i=1}^{n}e^{-\lambda x^{\beta}}$$

$$+(a-1)\sum_{i=1}^{n}\ln\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right) + (b-1)\sum_{i=1}^{n}\ln\left[1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right] \qquad (25)$$

Therefore, to obtain the MLE's of a, b, λ, α and β we find the first derivatives of the natural logarithm of the likelihood function with respect to a, b, λ, α and β and equating them to zero, we get the following five equations

$$\frac{\partial\ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\ln\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right) - (b-1)\sum_{i=1}^{n}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\left[1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{-1}\ln\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right),$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\ln\left[1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right],$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \frac{n[\alpha-1-\alpha\ln\alpha]}{\alpha(\alpha-1)\ln\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}e^{-\lambda x^{\beta}} + (a-1)\sum_{i=1}^{n}\left[\frac{(\alpha-1)(1-e^{-\lambda x^{\beta}})\alpha^{-e^{-\lambda x^{\beta}}} - (\alpha^{1-e^{-\lambda x^{\beta}}}-1)}{(\alpha-1)(\alpha^{1-e^{-\lambda x^{\beta}}}-1)}\right]$$

$$-a(b-1)\sum_{i=1}^{n}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\left[\frac{(\alpha-1)(1-e^{-\lambda x^{\beta}})\alpha^{-e^{-\lambda x^{\beta}}} - (\alpha^{1-e^{-\lambda x^{\beta}}}-1)}{(\alpha-1)(\alpha^{1-e^{-\lambda x^{\beta}}}-1)}\right]\left[1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{-1},$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}x_i^{\beta} + \ln\alpha\sum_{i=1}^{n}x_i^{\beta}e^{-\lambda x^{\beta}} + (a-1)\sum_{i=1}^{n}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}x_i^{\beta}e^{-\lambda x^{\beta}}\ln\alpha}{\alpha^{1-e^{-\lambda x^{\beta}}}-1}\right)$$

$$-a(b-1)\sum_{i=1}^{n}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}x_i^{\beta}e^{-\lambda x^{\beta}}\ln\alpha}{\alpha^{1-e^{-\lambda x^{\beta}}}-1}\right)\left[1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{-1}$$

and

$$\frac{\partial \ell}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln x_i-\lambda\sum_{i=1}^{n}x_i^{\beta}\ln x_i+\ln\alpha\sum_{i=1}^{n}\lambda x_i^{\beta}e^{-\lambda x^{\beta}}\ln x_i$$

$$+(a-1)\sum_{i=1}^{n}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}\lambda x_i^{\beta}e^{-\lambda x^{\beta}}\ln\alpha\ln x_i}{\alpha^{1-e^{-\lambda x^{\beta}}}-1}\right)$$

$$-a(b-1)\sum_{i=1}^{n}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}\lambda x_i^{\beta}e^{-\lambda x^{\beta}}\ln\alpha\ln x_i}{\alpha^{1-e^{-\lambda x^{\beta}}}-1}\right)\left[1-\left(\frac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1}\right)^{a}\right]^{-1}.$$

Then the maximum likelihood estimates of the parameters a, b, λ, α and β can be obtained by solving the above system of equations. No explicit form for these estimates, we use a numerical technique like Newton-Raphson method may be used to solve these non-linear equations.

For interval estimation and hypothesis tests on the model parameters, we require the observed information matrix whose elements are available with the corresponding author.

6. Applications

In this section, we provide two applications to two real data sets to prove the importance and flexibility of the KuAP-W distribution. The first data set refers to the actual taxes data. The data represent the monthly actual taxes revenue (in 1000 million Egyptian pounds) in Egypt from January 2006 to November 2010. The data are: 5.9, 20.4, 14.9, 16.2, 17.2,7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1,6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1,36, 8.5, 8, 9.2, 26.2, 21.9,16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7,8.6,12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8. These data were analyzed by Nassar and Nada (2011).

The second data set represents 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. This data set is obtained from Smith and Naylor (1987) and is has been analyzed by Afify et al. (2016a) for fitting the Weibull Fréchet distribution. The data are: 0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49,1.53,1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62,1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77,1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55,1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

For both data sets, we compare the fits of the KAPW model with some competitive models, namely: generalized Burr X Weibull (GBXW) (Aldahlan et al., 2018), exponentiated Weibull (EW) (Mudholkar et al., 1996), odd log-logistic exponentiated Weibull (Afify et al., 2018), alpha power Weibull (APW) (Nassar et al., 2017), transmuted complementary Weibull geometric (TCWG) (Afify et al., 2014), alpha logarithmic transformed Weibull (Nassar et al., 2018), Weibull Weibull (WW) (Abouelmagd et al., 2017) and Weibull distributions.

Table 3: MLEs (standard errors in parentheses), and the statistics $-\ell(\hat{\theta})$, KS and PV for the first data set

Distribution	Estimates					$-\ell(\hat{\theta})$	KS	PV
KAPW	323.173	0.8279	0.9761	16.5278	0.2607	187.887	0.0617	0.9779
$(\alpha,\beta,\lambda,a,b)$	(17.699)	(0.4678)	(1.0422)	(31.148)	(0.2665)			
GBXW	175.749	258.722	5.8501	0.0264		188.349	0.0631	0.9727
(α,β,a,b)	(159.08)	(433.25)	(0.8959)	(0.0079)				
EW	2813.00	0.2772	4.2640			188.241	0.0640	0.9686
(α,β,λ)	(11683)	(0.1420)	(3.5913)					
OLLEW	0.0721	0.1500	5.5155	7.0781		190.718	0.0727	0.9134
$(\alpha,\beta,\gamma,\theta)$	(0.1505)	(0.1433)	(6.9616)	(9.9837)				
APW	3432.25	0.8786	0.2811			192.019	0.1055	0.5266
(α,β,λ)	(4219.6)	(0.0934)	(0.0792)					
TCWG	0.9999	2.0179	0.6436	0.0538		195.706	0.1324	0.2518
$(\alpha,\beta,\lambda,\delta)$	(0.6072)	(0.3280)	(0.2337)	(0.0109)				
ALTW	0.4333	1.9431	0.0039			196.466	0.1228	0.3353
(α,β,λ)	(0.2708)	(0.1099)	(0.0014)					
WW	0.2636	77.5484	0.6699	0.0170		197.380	0.1432	0.1774
(α,β,a,b)	(4.9663)	(23.026)	(0.1279)	(0.0070)				
W	1.8403	0.0653				197.290	0.1431	0.1780
(β,λ)	(0.1711)	(0.0049)						

Tables 3 and 4 provide the MLEs of the model parameters, their corresponding standard errors (SEs) and the values of $-\ell(\hat{\theta})$, KS and PV for both data sets, respectively. The plots of the fitted KAPW PDF and other fitted densities, for the both data sets, are displayed in Figures 3 and 4, respectively. They reveal that the KAPW distribution provides the best fits and it can be considered very competitive model to other distributions with positive support for the two data sets.

Table 4: MLEs (standard errors in parentheses), and the statistics $-\ell(\hat{\theta})$, KS and PV for the second data set

Distribution	Estimates					$-\ell(\hat{\theta})$	KS	PV
KAPW	89.2219	5.4350	0.1260	0.4975	0.8040	12.150	0.0975	0.5864
$(\alpha,\beta,\lambda,a,b)$	(263.60)	(1.5714)	(0.0816)	(0.2354)	(0.8459)			
GBXW	0.4623	1.3915	0.0690	2.9125		14.565	0.1406	0.1653
(α,β,a,b)	(0.7270)	(0.8250)	(0.2522)	(2.7039)				
EW	0.6712	7.2844	0.0194			14.675	0.1462	0.1351
(α,β,λ)	(0.2209)	(1.4869)	(0.0210)					
OLLEW	1.9919	8.7488	0.3021	1.6872		14.024	0.1319	0.2223
$(\alpha,\beta,\gamma,\theta)$	(0.2971)	(3.9362)	(0.2664)	(0.7428)				
APW	10.8558	4.4836	0.1947			13.474	0.1224	0.3010
(α,β,λ)	(12.717)	(0.7626)	(0.1082)					
TCWG	0.0698	3.2035	-0.1380	0.8911		١٢,٠٣٠	0.0995	0.5598
$(\alpha,\beta,\lambda,\delta)$	(0.1140)	(0.9403)	(0.9303)	(0.1952)				
ALTW	22.528	4.4786	0.2549			13.575	0.1432	0.1507
(α,β,λ)	(42.543)	(0.7487)	(0.1913)					
WW	0.0278	3.1168	0.8617	1.0134		14.412	0.1373	0.1852
(α,β,a,b)	(0.0724)	(2.7740)	(0.5162)	(0.6708)				
W	5.7807	0.6142				15.206	0.1522	0.1078
(β,λ)	(0.5760)	(0.0139)						

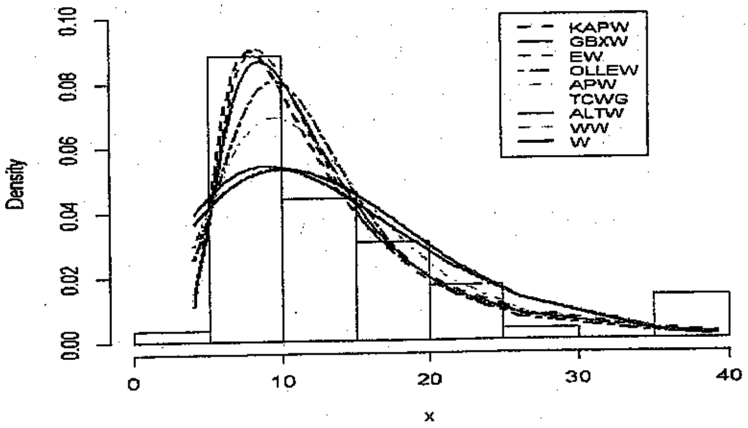

Figure 3: The estimated KuAP-W PDF and other estimated PDFs for the first data set

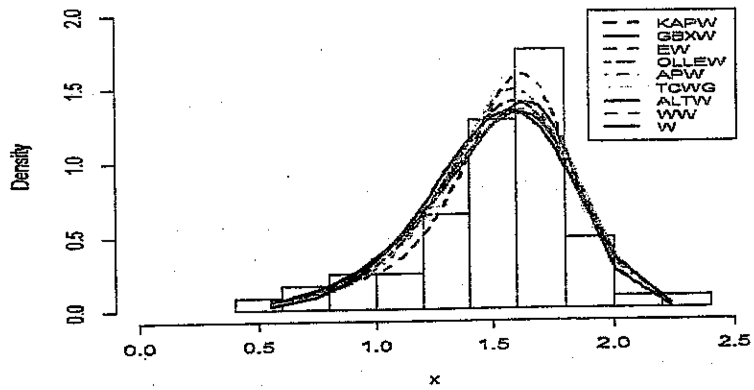

Figure 4: The estimated KuAP-W PDF and other estimated PDFs for the second data set

7. Concluding remarks

In this article, we introduce the Kumaraswamy alpha power-G (KuAP-G) family of distributions to extend the alpha power transform class defined by Mahdavi and Kundu (2015) and several other families. Based on the KAP-G class, we construct a

new four-parameter model called the Kumaraswamy alpha power Weibull (KuAP-W) distribution is proposed to serve as an alternative to many existing distributions. Although, it may not be guaranteed that the proposed model always yields better fits compared to existing models, it can serve in many cases as good alternative to them. Some mathematical properties such as the quantile and generating functions, ordinary and incomplete moments, residual and reversed residual lifes and order statistics are obtained. The model parameters are estimated by the maximum likelihood estimation method. Two applications to real data sets are presented to illustrate the flexibility of the KuAP-W model as compared to other existing models. The numerical results show that the KuAP-W model is better as compared to several others for these two data sets. We expect the utility of the newly proposed model in different fields especially in lifetime and reliability when the hazard rate is decreasing, increasing, bathtub or upside-down bathtub.

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